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DECOMPOSITION INTO PARTIAL FRACTIONS.

By Geo. F. Metzler.

I believe the following method of decomposition will commend itself to any who will use it enough to become familiar with its application.

A few examples will manifest its brevity.

(1) Let

$$\frac{1}{(x^2+1)(x^2+x+1)} \equiv \frac{ax+b}{y} + \frac{cx+d}{z},$$

where

$$y \equiv x^2+1, \quad \text{and} \quad z \equiv x^2+x+1.$$

Multiply both sides of (1) by y, then

$$\frac{1}{x^2+x+1} \equiv ax+b+\frac{y(cx+d)}{z},$$

which for values of x which make $y \equiv x^2+1=0$ becomes

$$ax+b \equiv \frac{1}{x} = \frac{x}{x^2} = \frac{x}{y-1} = -x;$$

therefore $a = -1$, $b = 0$.

Multiply (1) by z we have

$$\frac{1}{x^2+1} \equiv \frac{z(ax+b)}{y} + cx + d;$$

then when $z = 0$

$$cx+d \equiv \frac{1}{z-x} = \frac{1}{-x} = \frac{x+1}{-(x^2+x)} = -\frac{x+1}{z-1} = x+1;$$

$$\therefore c = 1 \quad \text{and} \quad d = 1.$$

(2) Again, let

$$\frac{x+1}{(x^2+1)^2(x-1)} \equiv \frac{a}{x-1} + \frac{bx+c+(dx+e)z}{(x^2+1)^2 \equiv z^2}.$$

Then

$$a = \frac{x+1}{(x^2+1)^2} = \frac{2}{4} = \frac{1}{2},$$

when $x - 1 = 0$. Therefore $a = 1/2$.

Multiply (2) by z^2 and express the result in ascending powers of z; then

$$bx + c + (dx + e)z \equiv \frac{x+1}{x-1} - \frac{az^2}{x-1} = \frac{x^2+2x+1-a(x+1)z^2}{x^2-1}$$

$$= \frac{2x+z-a(x+1)z^2}{-2+z} = -x - \frac{x+1}{2}z.$$

$$\therefore\ b = -1, \quad c = 0, \quad d = -\tfrac{1}{2}, \quad e = -\tfrac{1}{2}.$$

This depends on the theorem: If an equation is an identity in x, it will become an identity in z, when it is transformed into a function of z, where z is a function of x.

Then, as we know the coefficients of z^2 and higher powers of z will equal zero, the term containing a might have been omitted in finding $bx + c + (dx + e)z$. Thus

$$bx + c + (dx + e)z \equiv \frac{x+1}{x-1} \equiv \frac{x^2+2x+1}{x^2-1} = \frac{2x+z}{-2+z} = \text{etc.}$$

It will be noticed that the numerator and denominator of $(x+1)/(x-1)$ was multiplied by $x+1$, the object being to make the denominator a function of z with coefficients independent of x.

This auxiliary factor is found thus

$$\frac{z}{x-1} = \frac{x^2+1}{x-1} = x + 1 + \frac{2}{x-1}; \ \therefore 2 + (x+1)(x-1) = z$$

or

$$(x+1)(x-1) = z - 2,$$

from which x has disappeared; then both numerator and denominator are arranged in ascending powers of z and the division performed.

(3) Let

$$\frac{x^2}{(x^2-1)(x^2+2)} = \frac{a}{x-1} + \frac{b}{x+1} + \frac{cx+d}{x^2+2}$$

a and b are easily found as above.

Multiply by $x^2 + 2$ and taking values that make $x^2 + 2 = 0$

$$cx + d = \frac{x^2}{x^2 - 1} = \frac{-2}{-3} = \frac{2}{3} \cdots.$$

(4) Let

$$\frac{x^2 + 1}{(x - 1)^4(x^3 + 1)} \equiv \frac{a}{y} + \frac{bx + c}{z} + \frac{d + ew + fw^2 + gw^3}{w^4},$$

where

$$y = x + 1, \quad z = x^2 - x + 1, \quad w = x - 1.$$

Multiply by z and use values that make

$$z = 0 \quad \text{or} \quad x^2 - x + 1 = 0,$$

$$bx + c = \frac{x^2 + 1}{(x^2 - 2x + 1)^2(x + 1)} = \frac{z + x}{(z - x)^2(x + 1)}$$

$$= \frac{x}{x^2(x + 1)} = \frac{1}{x^2 + x} = \frac{1}{z + 2x - 1} = \frac{1(2x - 1)}{(2x - 1)(2x - 1)}$$

$$= \frac{2x - 1}{4z - 3} = \frac{2x + 1}{-3}.$$

The auxiliary factor $2x - 1$, which multiplies both numerator and denominator, is found by dividing $4z$ by $2x - 1$.

Now multiply by w and expand in ascending powers of w we have

$$d + ew + fw^2 + gw = \frac{x^2 + 1}{x^3 + 1} \cdots$$

powers of w higher than the third have coefficients equal to zero and may be omitted

$$d + ew + fw^2 + gw^3 = \frac{x^2 + 1}{x^3 + 1} = \frac{2 + 2w + w^2}{2 + 3w + 3w^2 + w^3}$$

$$= 1 - \frac{w}{2} - \frac{w^2}{4} + \frac{5w}{8}.$$

In this no auxiliary factor was needed because $x - 1$ is so simple. But if the denominator were $ax^2 + bx + c$ and the variable $w = dx^2 + rx + s$, then the auxiliary factor would be

$$d^2(ax^2 + bx + c) + (az - bd)\left(\frac{dw}{dx}\right),$$

or

$$ad^2x^2 + (2az - bd)x + cd^2 + z(az - bd),$$

and this multiplied by $ax^2 + bx + c$ gives

$$a^2w^2 + [za(cd - as) - b(bd - ar)]w + (cd - as)^2 + (cr - bs)(ar - bd)$$

which is the denominator required from which x has disappeared. This shows that the auxiliary factor may be found by using undetermined coefficients. I have never had occasion to use one so cumbersome.

Now we undertake a problem of some difficulty, in which 14 unknown quantities are to be found. We assume a knowledge of multiplication and division by detached coefficients and of how to reduce any rational function of x to a function of y where y is a rational function of x.

(5) Let

$$\frac{x^2+2}{z^2 \cdot w^4 \cdot v^2} = \frac{a+bz}{z^2} + \frac{cx+d+(ex+f)w+(gx+h)w^3+(ix+j)w^3}{w^4} + \frac{kx+l+(mx+n)v}{v^2},$$

where

$$z = x - 2, \quad w = x^2 + 1 \quad \text{and} \quad v = x^2 - x + 1.$$

Multiply by z^2 and omit z^2 and higher powers.

$$a + bz = \frac{x^2+2}{w^4 \cdot v^2} = \frac{(z+2)^2 + 2}{(z^2 + 4z + 5)^4(z^2 + 3z + 3)^2}$$

$$= \frac{6 + 4z + + \cdots}{9 \cdot 5^3(5 + 26z + +)} = \frac{30 - 136z}{9 \times 5^5}.$$

There is no need of an auxiliary factor, when z is of first degree in x.

Multiply by v^2 and omit all powers of v higher than the first.

$$kx + l + (mx + n)v = \frac{x^2+2}{z^2 \cdot w^4} = \frac{(x+1)^2(x^2+2)(x^2-2x+2)^4}{z^2(x+1)^2(w^4)(x^2-2x+2)^4}$$

$$= \frac{(x+1)^2(x^2+2)(x^2-2x+2)^4}{(v-3)^2(v^2+1)^4 = 9 - 6v + + \cdots},$$

$$\begin{array}{rrrrrrrrrrrrrr}
(x^2-2x+2)^4 = & x^8 & -8x^7 & +32x^6 & -80x^5 & +136x^4 & -160x^3 & +128x^2 & -64x & +16 & & & & \\
 & 1 & -8 & +32 & -80 & +136 & -160 & +128 & -64 & +16 & & & & \\
 & & 2 & -16 & +64 & -160 & +272 & -320 & +256 & -128 & +32 & & & \\
 & & & +1 & -8 & +32 & -80 & +136 & -160 & +128 & -64 & +16 & & \\
\hline
(x^2-2x+2)^4(x+1)^2 = & 1 & -6 & +17 & -24 & +8 & +32 & -56 & +32 & +16 & -32 & +16 & & \\
 & & & 2 & -12 & +34 & -48 & +16 & +64 & -112 & +64 & +32 & -64 & +32 \\
\hline
(x^2+2x+2)^4(x+1)^2(x^2+2) = & 1 & -6 & +19 & -36 & +42 & -16 & -40 & +96 & -96 & +32 & +48 & -64 & +32
\end{array}$$

Reducing this to a function of v and omitting powers higher than the first

$$\begin{array}{r|rrrrrrrrrrrrr}
-1 & 1 & -6 & +19 & -36 & +42 & -16 & -40 & +96 & -96 & +32 & +48 & -64 & +32 \\
+1 & & & -1 & +5 & -13 & +18 & -11 & -13 & +38 & -45 & +13 & +26 & -35 \\
 & & +1 & -5 & +13 & -18 & +11 & +13 & -38 & +45 & -13 & -26 & +35 & \\
\hline
 & 1 & -5 & +13 & -18 & +11 & +13 & -38 & +45 & -13 & -26 & +35 & \multicolumn{2}{|l}{-3x-3} \\
 & & & -1 & +4 & -8 & +6 & +3 & -16 & +19 & -10 & -16 & & \\
 & & +1 & -4 & +8 & -6 & -3 & +16 & -19 & +10 & +16 & & & \\
\hline
 & 1 & -4 & +8 & -6 & -3 & +16 & -19 & +10 & +16 & \multicolumn{2}{|l}{-20x+19} & &
\end{array}$$

Divide

$$\frac{-3(x+1) + (-20x+19)v}{9-6v} = -\frac{x+1}{3} + \frac{17-22x}{9}v$$

$$= kx + l + (mx + n)v.$$

Multiply by w^4

$$cx + d + (ex + f)w + (gx + h)w^2 + (ix + j)w^3$$

$$+ \frac{x^2+2}{z^2 \cdot v^2} = \frac{(x+2)^2(x^2+2)(x^2+x+1)^2}{(x^2-4)^2(w^2-x^2)^2},$$

$$(x^2-4)^2(w^2-x^2)^2 = (w-5)^2(w^2-w+1)^2$$

$$= 25 - 60w + 96w^2 - 82w^3,$$

$$
\begin{array}{rrrrrrrrrr}
(x^2+x+1)^2=x^4+2x^3+3x^2+2x+1, & & & & & & & & & \\
 & 1+ & 2+ & 3+ & 2+ & 1 & & & & \\
 & & 4+ & 8+ & 12+ & 8+ & 4 & & & \\
 & & & 4+ & 8+ & 12+ & 8+ & 4 & & \\
\hline
 & 1+ & 6+ & 15+ & 22+ & 21+ & 12+ & 4 & & \\
 & & & 2+ & 12+ & 30+ & 44+ & 42+ & 24+ & 8 \\
\hline
(x^2+x+1)^2(x^2+2)(x+2)^2= & 1+ & 6+ & 17+ & 34+ & 51+ & 56+ & 46+ & 24+ & 8
\end{array}
$$

Reducing to a function of w,

$$
\begin{array}{r|rrrrrrrrr}
-1 & 1+ & 6+ & 17+ & 34+ & 51+ & 56+ & 46+ & 24+ & 8 \\
0 & & & -1- & 6- & 16- & 28- & 35- & 28- & 11 \\
 & & +0+ & 0+ & 0 & & & & & \\
\hline
 & 1+ & 6+ & 16+ & 28+ & 35+ & 28+ & 11- & 4x- & 3 \\
 & & -0- & 1- & 6- & 15- & 22- & 20 & & \\
\hline
 & 1+ & 6+ & 15+ & 22+ & 20+ & 6x+ & 9 & & \\
 & & & -1- & 6- & 14 & & & & \\
\hline
 & 1+ & 6+ & 14+ & 16+ & 6 & & & & \\
 & & & -1 & & & & & & \\
\hline
 & 1+ & 6+ & 13 & & & & & &
\end{array}
$$

Then

$$\frac{-(4x+3)+(6x+9)w+(16x+6)w^2+(6x+13)w^3}{25-60w+96w^2-82w^3}$$

$$=-\frac{(4x+3)}{25}-\frac{18x+81}{125}w-\frac{568x-534}{625}w^2+\frac{7654x+1763}{3165}w^3$$

$$\equiv cx+d+(ex+f)w+(gx+h)w^2+(ix+j)w^3.$$

In like manner

$$\frac{x^2-x+1}{(x^2-x+2)^2(x^2+x+3)^3}=\frac{29+10x}{1331(x^2-x+2)^2}+\frac{22x-60}{1331(x^2-x+2)}$$

$$+\frac{242(x+1)}{1331(x^2+x+3)^3}+\frac{(4x+17)11}{1331(x^2+x+3)^2}-\frac{22x+16}{1331(x^2+x+3)},$$

(7) $$\frac{x^5-4x^4-25x^3+68x^2-13x-38}{(x-1)(x-2)^3(x^2+x+2)}=\frac{a}{x-1}$$

$$+\frac{b+cz+dz^2}{(x-2)^3=z^3}+\frac{ex+g}{y}\qquad\left\{\begin{array}{l}y=x^2+x+2,\\ z=x-2.\end{array}\right.$$

Then

$$b + cz + dz^2 = \frac{x^5 - x^4 - 25x^3 + 68x^2 - 13x - 38}{(x - 1)(x^2 + x + 2)},$$

$$\begin{array}{r|rrrrrr}
2 & 1 & -1 & -25 & +68 & -13 & -38 \\
 & & +2 & +2 & -46 & +44 & +62 \\ \cline{2-7}
 & 1 & +1 & -23 & +22 & +31 & +24 \\
 & & +2 & +6 & -34 & -24 & \\ \cline{2-6}
 & 1 & +3 & -17 & -12 & +7 & \\
 & & +2 & +10 & -14 & & \\ \cline{2-5}
 & 1 & +5 & -7 & -26 & &
\end{array}$$

$$\begin{array}{r|rrrrl}
 & 1 & +1 & +2 & & \\
 & & -1 & -1 & -2 & \\ \hline
2 & 1 & +0 & +1 & -2 & = (x^2 + x + 2)(x - 1) \\
 & & +2 & +4 & +10 & \\ \cline{2-5}
 & 1 & +2 & +5 & +8 & \\
 & & +2 & +8 & & \\ \cline{2-4}
 & 1 & +4 & +13 & & \\
 & & +2 & & & \\ \cline{2-3}
 & 1 & +6 & & &
\end{array}$$

$$\frac{24 + 7z - 26z^2 + \cdots}{8 + 13z + 6z^2} = 3 - 4z + z^2 \equiv b + cz + dz^2$$

Multiply by y

$$ex + g = \frac{x^5 - x^4 - 25x^3 + 68x^2 - 13x - 38}{(x - 1)(z^3)}$$

$$\begin{array}{r|rrrrrr}
-2 & 1 & -1 & -25 & +68 & -13 & -38 \\
-1 & & & -2 & +4 & +50 & -194 \\
 & & -1 & +2 & +25 & -97 & \\ \hline
 & 1 & -2 & -25 & +97 & -60 & -232
\end{array}$$

$$\begin{array}{r|rrrrr}
\multicolumn{1}{r}{z^3(1 - x) =} & 1 & -7 & +18 & -20 & +8 \\
-2 & & & -2 & +16 & -48 \\
-1 & & -1 & +8 & -24 & \\ \cline{2-6}
 & 1 & -8 & +24 & -28 & -40
\end{array}$$

$$\frac{-232 - 60x}{-40 - 28x} = \frac{(58 + 15x)(7x - 3)}{(10 + 7x)(7x - 3)}$$

$$= \frac{105x^2 + 361x - 174}{49x^2 + 49x - 30} = \frac{105y + 256x - 384}{49y - 128},$$

when powers of y are omitted this

$$= 3 - 2x \equiv g + ex, \quad 7x - 3 = \frac{49y}{7x + 10}.$$

In this case the auxiliary factor was introduced after the fraction was transformed to powers of y.

If $ax + b$ is a factor in the denominator the auxiliary factor will be $(a^2y/ax + \mathrm{b}) \ldots$, if y contains x^2.

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